Rationality of moduli of vector bundles on curves

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1 Introduction

Let C be a smooth projective curve of genus g over an algebraically closed field k. Let $\mathfrak{M}_{r,d}$ be the moduli space of stable vector bundles of rank r and degree d over C. This is a smooth quasi-projective variety of dimension $r^2(g-1)+1$, which is projective when r and d are coprime. Up to isomorphism, it depends only on the congruence class of d mod r. The rank 1 case $\mathfrak{M}_{1,d}$ is isomorphic to the Jacobian $J(C)$ and every moduli space comes equipped with a determinant map $\det\colon \mathfrak{M}_{r,d} \to \mathfrak{M}_{1,d}$ whose fibre over L is $\mathfrak{M}_{r,L}$, the moduli space of bundles with fixed determinant L.

The goal of this paper is to describe these moduli spaces in the birational category, that is, to describe their function fields. We shall prove the following result.

Theorem 1.1. *The moduli space $\mathfrak{M}_{r,d}$ is birational to $\mathfrak{M}_{h,0} \times \mathbb{A}^{(r^2-h^2)(g-1)}$, where $h = \operatorname{hcf}(r,d)$.*

In other words, there is a dominant rational map $\mu\colon \mathfrak{M}_{r,d} \dashrightarrow \mathfrak{M}_{h,0}$ whose generic fibre is rational. We shall observe that this map restricts to a map between fixed determinant moduli spaces (not necessarily with the same determinant) and so, in the case when r and d are coprime, we obtain the following long believed corollary, which has been proved in special cases ([5],[6],[1]).

Theorem 1.2. *If L is a line bundle of degree d coprime to r, then $\mathfrak{M}_{r,L}$ is a rational variety.*

To ease the discussion, we use the following terminology to describe the relationship between $\mathfrak{M}_{r,d}$ and $\mathfrak{M}_{h,0}$. An irreducible algebraic variety X is

birationally linear over another irreducible algebraic variety Y if there exists a dominant rational map $\phi\colon X \dashrightarrow Y$ whose generic fibre is rational, that is, the function field $k(X)$ is purely transcendental over the function field $k(Y)$. Such a map ϕ will also be called *birationally linear*.

What we shall actually prove is a stronger statement that the map μ is birationally linear *and* preserves a suitable Brauer class. More precisely, for each type (r, d) with $\mathrm{hcf}(r, d) = h$, the moduli space $\mathfrak{M}_{r,d}$ carries a Brauer class $\psi_{r,d}$ for its function field, represented by a central simple algebra of dimension h^2, and the map μ has the property that $\mu^*(\psi_{h,0}) = \psi_{r,d}$. This strengthening of the statement is the key to the proof, because it enables an induction on the rank r.

In section 2, we construct an open dense subvariety of $\mathfrak{M}_{r,d}$ as a quotient space of a suitable variety $X_{r,d}$ by a generically free action of PGL_h where $h = \mathrm{hcf}(r, d)$. This arises because we are able to show that a general vector bundle E of type (r, d) arises as a quotient of a particular bundle F^h in a unique way so that we can take $X_{r,d}$ to be a suitable open subvariety of $\mathrm{Quot}(F^h, r, d)$ on which PGL_h acts in the natural way. We also arrange in this section that the kernel of the surjection from F^h to E should have smaller rank (at least in the case where $h \neq r$) and this induces a rational map from $\mathfrak{M}_{r,d}$ to $\mathfrak{M}_{r_1,d_1}$ for some type (r_1, d_1) where $r_1 < r$. After this, in section 3, we show how this description of $\mathfrak{M}_{r,d}$ as a quotient space for a generically free action of PGL_h allows us to associate a Brauer class to its function field and we use this Brauer class to describe birationally the rational map from $\mathfrak{M}_{r,d}$ to $\mathfrak{M}_{r_1,d_1}$ in terms of "twisted Grassmannian varieties". In section 4, we use parabolic moduli spaces which give us other "twisted Grassmannian varieties" which we may choose to be twisted "in the same way" as our map between moduli spaces above. In the final section, we put these various results together to construct a birationally linear rational map from $\mathfrak{M}_{r,d}$ to $\mathfrak{M}_{h,0}$.

2 The first step

The purpose of this section is to show that the general bundle E of rank r and degree d may be constructed as a quotient of F^h, where F is a *fixed* bundle of an appropriate type and $h = \mathrm{hcf}(r, d)$.

This will enable us to define the Brauer class on $\mathfrak{M}_{r,d}$ that will be the focus of most of the paper. Furthermore, we will see that the kernel of the

quotient map $q : F^h \to E$ is also general so that we may define a dominant rational map from $\mathfrak{M}_{r,d}$ to $\mathfrak{M}_{r_1,d_1}$, where $r_1 < r$ when r does not divide d. This will be the basis of the inductive construction of the birationally linear map to $\mathfrak{M}_{h,0}$.

We will say that a vector bundle E of rank r and degree d has 'type' (r, d). For E of type $\alpha = (r_E, d_E)$ and F of type $\beta = (r_F, d_F)$, we write

$$\begin{aligned}\chi(F, E) &= \hom(F, E) - \operatorname{ext}(F, E)\\ &= r_F d_E - r_E d_F - r_E r_F (g - 1) = \chi(\beta, \alpha),\end{aligned}$$

where $\hom(F, E) = \dim \operatorname{Hom}(F, E)$ and $\operatorname{ext}(F, E) = \dim \operatorname{Ext}(F, E)$. The middle equality is the Riemann-Roch Theorem.

We start with a lemma about the nature of generic maps between generic vector bundles. The proof is closely based on the proof given by Russo & Teixidor ([7] Theorem 1.2) that the tensor product of generic bundles is not special; a result originally due to Hirschowitz.

Lemma 2.1. *Let E, F be generic vector bundles of fixed types. Suppose that there exists a non-zero map $\phi\colon F \to E$ and take ϕ to be a generic such map. Then* $\operatorname{ext}(F, E) = 0$ *and ϕ has maximal rank. If $r_E \neq r_F$, then* $\operatorname{coker} \phi$ *is torsion-free; in particular, if $r_E < r_F$, then ϕ is surjective and if $r_E > r_F$ then ϕ is injective.*

Proof. Let $[\phi]$ denote the homothety class of ϕ in $\mathbb{P}(\operatorname{Hom}(F, E))$. Then the triple $(F, E, [\phi])$ depends on

$$p_0 = 1 - \chi(F, F) + 1 - \chi(E, E) + \hom(F, E) - 1$$

parameters (cf. [2] Section 4). Let $I = \operatorname{im} \phi$, $K = \ker \phi$, $Q = \operatorname{coker} \phi$ and T be the torsion subsheaf of Q. Further, let $Q' = Q/T$ and I' be the inverse image of T in E. Thus we have three short exact sequences

$$0 \to K \to F \to I \to 0 \tag{2.1}$$

$$0 \to I \to E \to Q \to 0 \tag{2.2}$$

$$0 \to I' \to E \to Q' \to 0 \tag{2.3}$$

in which all terms except Q are vector bundles. The triple $(F, E, [\phi])$ is determined by the first and last sequences (up to homothety) and a map

$t\colon I \to I'$ whose cokernel is T. The triple $(I, I', [t])$ depends on $1 - \chi(I, I) + r_I d_T$ parameters and so the whole configuration depends on at most

$$\begin{aligned} p_1 = 1 - \chi(K, K) + 1 - \chi(I, I) + r_I d_T + 1 - \chi(Q', Q') \\ + \operatorname{ext}(I, K) - 1 + \operatorname{ext}(Q', I') - 1 \end{aligned}$$

parameters. Now, E and F are stable, so $\hom(I, K) = \hom(Q', I') = 0$ and hence $\operatorname{ext}(I, K) = -\chi(I, K)$ and $\operatorname{ext}(Q', I') = -\chi(Q', I')$. Furthermore $\chi(Q', Q') = \chi(Q, Q)$ and

$$\chi(Q', I') = r_Q(d_I + d_T) - r_I(d_Q - d_T) - r_Q r_I(g - 1) = \chi(Q, I) + r_E d_T$$

Hence, using the bilinearity of χ in short exact sequences, we get

$$p_1 = 1 - \chi(F, K) - \chi(Q, E) - \chi(I, I) - r_Q d_T$$

But now $p_0 \leq p_1$ and so

$$\begin{aligned} \hom(F, E) &\leq \chi(F, I) + \chi(I, E) - \chi(I, I) - r_Q d_T \\ &= \chi(F, E) - \chi(K, Q) - r_Q d_T \end{aligned}$$

and hence

$$\operatorname{ext}(F, E) \leq -\chi(K, Q) - r_Q d_T \tag{2.4}$$

A simple dimension count (cf. [3] Lemma 2.1) shows that, for general E and F to appear in sequences (2.1) and (2.3), it is necessary that $\chi(K, I) \geq 0$ and $\chi(I', Q') \geq 0$. In other words,

$$\begin{aligned} r_K d_I - r_I d_K \geq r_K r_I (g - 1) \\ r_I d_Q - r_Q d_I \geq r_I r_Q (g - 1) + r_E d_T \end{aligned}$$

and hence

$$\begin{aligned} \chi(K, Q) &= \big(r_K(r_I d_Q - r_Q d_I) + r_Q(r_K d_I - r_I d_K) - r_K r_I r_Q (g - 1)\big)/r_I \\ &\geq r_K r_Q (g - 1) + d_T (r_K r_E / r_I) \end{aligned}$$

Thus, subsituting this into (2.4), we finally deduce that

$$\operatorname{ext}(F, E) \leq -r_K r_Q (g - 1) - d_T (r_Q + r_K r_E / r_I)$$

This is only possible if (i) $\operatorname{ext}(F, E) = 0$, (ii) $r_K = 0$ or $r_Q = 0$ and (iii) unless $r_Q = r_K = 0$, we also have $d_T = 0$. But (ii) means that r_I has maximal rank and then (iii) means that $\operatorname{coker} \phi$ is torsion-free, unless $r_E = r_F$. $\square$

We shall also use the following lemma which may be thought of as a generalisation of the result that any (bounded) family of bundles on a curve may be extended to an irreducible family (cf. [4] Proposition 2.6).

Lemma 2.2. *Let $\{\mathcal{G}_x : x \in X\}$ be an irreducible family of vector bundles over C and let $\{\mathcal{E}_y : y \in Y\}$ be any family of vector bundles over C of fixed type. Then there exists an irreducible family of extensions of vector bundles,*

$$\{0 \to \mathcal{G}'_z \to \mathcal{F}'_z \to \mathcal{E}'_z \to 0 : z \in Z\}$$

such that every vector bundle $\mathcal{G}'_z$ is isomorphic to some $\mathcal{G}_x$ and every extension $0 \to \mathcal{G}_x \to \mathcal{F} \to \mathcal{E}_y \to 0$ is isomorphic to one in this family.

Proof. After twisting by a suitable line bundle of positive degree, we may assume that $\operatorname{Ext}^1(\mathcal{O}, \mathcal{G}_x) = 0$, for all $x \in X$, and that every $\mathcal{E}_y$ is generated by global sections. Suppose that each $\mathcal{E}_y$ is of type (n, d). Extending the usual dimension counting argument in the Grassmannian $\mathrm{G}\textsc{r}(n, H^0(\mathcal{E}_y))$, we may choose n sections of $\mathcal{E}_y$ so that the induced map $\rho : \mathcal{O}^n \to \mathcal{E}_y$ is an isomorphism of the fibres at the general point of C and drops rank by only 1 at other points. Thus the cokernel of ρ is the structure sheaf $\mathcal{T}_\xi$ of a subscheme ξ of degree d in C, that is, $\mathcal{E}_y$ is an extension of $\mathcal{T}_\xi$ on top of $\mathcal{O}^n$.

The parameter space of such subschemes ξ is the d-fold symmetric product $C^{(d)}$, which is an irreducible algebraic variety and which carries a universal family $\mathcal{T}$. Since $\mathcal{T}_\xi$ is torsion, $\operatorname{Hom}(\mathcal{T}_\xi, \mathcal{G}_x \oplus \mathcal{O}^n) = 0$ for all $\xi \in C^{(d)}$ and all $x \in X$. Hence there is a vector bundle $\lambda : Z \to X \times C^{(d)}$ whose fibre above the point (x, ξ) is $\operatorname{Ext}(\mathcal{T}_\xi, \mathcal{G}_x \oplus \mathcal{O}^n)$ and this carries a tautological family of extensions

$$\{0 \to \mathcal{G}_{\pi_1\lambda(z)} \oplus \mathcal{O}^n \to \mathcal{F}'_z \to \mathcal{T}_{\pi_2\lambda(z)} \to 0 : z \in Z\}.$$

Letting $\mathcal{G}'_z = \mathcal{G}_{\pi_1\lambda(z)}$ and $\mathcal{E}'_z = \mathcal{F}'_z/\mathcal{G}'_z$, we may replace Z by the non-empty open set on which $\mathcal{F}'_z$ and $\mathcal{E}'_z$ are vector bundles and obtain the required irreducible familty of extensions of vector bundles. To see that every possible extension of $\mathcal{E}_y$ on top of $\mathcal{G}_x$ occurs note that every such extension has a 3 step filtration with T_ξ on top of $\mathcal{O}^n$ on top of $\mathcal{G}_x$. But, since $\operatorname{Ext}^1(\mathcal{O}, \mathcal{G}_x) = 0$, the extension at the bottom of this filtration splits and so it is simply an extension of T_ξ on top of $\mathcal{G}_x \oplus \mathcal{O}^n$. $\square$

Using these lemmas, we have the following result.

Proposition 2.3. *For any type $\alpha = (r, d)$, let $h = \mathrm{hcf}(r, d)$. Then there is a unique type $\beta = (s, e)$ satisfying*

(i) $\chi(\beta, \alpha) = h$,

(ii) $r/h < s < 2r/h$, *if* $h < r$, *or* $s = 2$, *if* $h = r$.

Then, there exists a vector bundle F of type β such that for a general E of type α,

(iii) $\hom(F, E) = h$ *and* $\mathrm{ext}(F, E) = 0$,

(iv) the natural map $\varepsilon_F(E)\colon \mathrm{Hom}(F, E) \otimes_k F \to E$ *is surjective,*

(v) the bundle $E_1 = \ker \varepsilon_F(E)$ *is general and has* $ext(E_1, F) = 0$.

Proof. To solve (i) we simply need to solve $sd - tr = h$ and set $e = t - (g-1)s$. Given one solution (s, t), the complete set of solutions is $\{(s, t) + k(r/h, d/h) : k \in \mathbb{Z}\}$ which contains precisely one solution in the range (ii). Part (iii) is provided by Lemma 2.1.

For the main part of the proof, the first step is to construct a short exact sequence

$$0 \to E_1 \to F^h \to E \to 0 \tag{2.5}$$

with E of type α, F of type β and $\mathrm{Ext}(E_1, F) = 0 = \mathrm{Ext}(F, E)$.

First suppose that $h = 1$. Then Lemma 2.1 implies that for generic F and E we have $\mathrm{Ext}(F, E) = 0$ and, since $r < s$, the generic map is surjective. Let $F' \to E'$ be a particular choice of such generic bundles and map and let E_1' be the kernel. At this stage, we have $\mathrm{Ext}(F', E') = 0$, but may not have $\mathrm{Ext}(E_1', F') = 0$. On the other hand,

$$\chi(\beta - \alpha, \beta) = \chi(\beta, \alpha) + \chi(\beta - \alpha, \beta - \alpha) - \chi(\alpha, \alpha) \geq \chi(\beta, \alpha) = 1$$

since $s - r \leq r$. Hence, Lemma 2.1 also implies that, for generic E_1 and F of types $\beta - \alpha$ and β respectively, $\mathrm{Ext}(E_1, F) = 0$ and, since $s - r < s$, the generic map is an injection of vector bundles. Let $E_1'' \to F''$ be a particular choice of such generic bundles and map and let E'' be its cokernel. This time, we have $\mathrm{Ext}(E_1'', F'') = 0$, but may not have $\mathrm{Ext}(F'', E'') = 0$.

But now we may include E_1' and E_1'' in an irreducible family $\{\mathcal{E}_{1,x} : x \in X\}$ by [4] Proposition 2.6. Then, by Lemma 2.2, there is an irreducible family of extensions

$$\{0 \to \mathcal{G}_z' \to \mathcal{F}_z' \to \mathcal{E}_z' \to 0 : z \in Z\}$$

which includes both $0 \to E_1' \to F' \to E' \to 0$ and $0 \to E_1'' \to F'' \to E'' \to 0$. Hence we may choose for (2.5) a general extension in this family and both Ext groups will vanish as required.

For an arbitrary value of h, we may obtain a sequence of the form (2.5) by taking the direct sum of h copies of one for $\alpha = \alpha/h$.

For the second step, suppose that we have a sequence of the form (2.5). By [4] Proposition 2.6, we may include E_1 in an irreducible family $\{\mathcal{E}_{1,x} : x \in X\}$, whose generic member is general and for which every member satisfies $\operatorname{ext}(\mathcal{E}_{1,x}, F) = 0$. There is then a vector bundle $\lambda : Y \to X$ whose fibre at x is $\operatorname{Hom}(\mathcal{E}_{1,x}, F^h)$ and over Y there is a tautological map $f_y : \mathcal{E}_{1,\lambda(y)} \to F^h$. Replacing Y by the non-empty open set on which f_y is injective, we have $\mathcal{E}_y = \operatorname{coker} f_y$ of type α. We may further replace Y by the non-empty open set on which $\operatorname{ext}(F, \mathcal{E}_y) = 0$.

Now observe that the homomorphism $F^h \to \mathcal{E}_y$ must be isomorphic to $\varepsilon_F(\mathcal{E}_y)\colon \operatorname{Hom}(F, \mathcal{E}_y) \otimes_k F \to \mathcal{E}_p$, because a linear dependence between the h components of the homomorphism from F^h to $\mathcal{E}_y$ would imply that F is a summand of the kernel, which would contradict $\operatorname{ext}(\mathcal{E}_{1,x}, F) = 0$.

If we consider just the family $\{\mathcal{E}_y : y \in Y\}$, then, as before, we may include this in an irreducible family $\{\mathcal{E}_z : z \in Z\}$, whose generic member is general and such that $\operatorname{ext}(F, \mathcal{E}_z) = 0$ and $\operatorname{hom}(F, \mathcal{E}_z) = h$ for every $z \in Z$. Hence the kernel $\mathcal{E}_{1,z}'$ of the homomorphism $\varepsilon_F(\mathcal{E}_z)\colon \operatorname{Hom}(F, \mathcal{E}_z) \otimes_k F \to \mathcal{E}_z$ is general and satisfies $\operatorname{ext}(\mathcal{E}_{1,z}', F) = 0$, because this was already true over Y. Thus we have all the properties we require. $\square$

Proposition 2.3 shows that we have a dominant rational map

$$\lambda_F\colon \mathfrak{M}_{r,d} \dashrightarrow \mathfrak{M}_{r_1,d_1} : [E] \mapsto [E_1],$$

where $E_1 = \ker \varepsilon_F(E)$ has type $(r_1, d_1) = h(s, e) - (r, d)$. One may immediately check the following.

Lemma 2.4. *The type (r_1, d_1) of E_1 satisfies*

(i) if $h < r$, then $r_1 < r$,

(ii) $h_1 = \mathrm{hcf}(r_1, d_1)$ *is divisible by* h*,*

(iii) $\det(E_1) \cong \det(F)^h \det(E)^{-1}$*.*

The proof of Proposition 2.3 shows that the fibre of λ_F above a closed point $[E_1]$ is birationally the Grassmannian of h-dimensional subspaces of $\mathrm{Hom}(E_1, F)$. However, this bundle of Grassmannians may be 'twisted', that is, it may not be locally trivial in the Zariski topology. In fact, it will fail to be locally trivial whenever $h \neq 1$ and will not be birationally linear whenever $h_1 \neq h$, but we will be able to measure how twisted it is using a Brauer class on $\mathfrak{M}_{r_1,d_1}$ and then compare λ_F to another Grassmannian bundle with the same twisting, but smaller fibres, to construct inductively our birationally linear map.

In fact, Proposition 2.3 also provides us with the way of constructing this Brauer class, because it yields a description of $\mathfrak{M}_{r,d}$ as a quotient of an open set in the quot scheme $\mathrm{Quot}(F^h, r, d)$ by PGL_h. We describe this in detail in the next section.

3 Brauer classes and free PGL actions

In this section, we collect a number of results about free actions of the projective general linear group PGL, which allow us to define and compare the Brauer classes we are interested in.

Recall that the Brauer group of a field k may be described as consisting of classes represented by central simple algebras A over the field and that $[A_1] = [A_2]$ in the Brauer group if and only if A_1 and A_2 are Morita equivalent or equivalently $A_1^o \otimes A_2$ is isomorphic to $M_n(k)$ for a suitable integer n where A^o is the opposite algebra to A. This is equivalent to saying that there is an A_1, A_2 bimodule of dimension n where $n^2 = \dim A_1 \dim A_2$. The product in the Brauer group is induced by the tensor product of algebras.

The reader may wish to consult [8] for further discussion of the Brauer group and central simple algebras.

Definition 3.1. Let X be an affine algebraic variety on which the algebraic group PGL_n acts freely. Over the quotient variety X/PGL_n there is a bundle of central simple algebras $M_n(k) \times^{PGL_n} X$ of dimension n^2. At the generic point, this is a central simple algebra over the function field $k(X/PGL_n)$ and hence defines a class in the Brauer group of $k(X/PGL_n)$. We shall denote this class by $\mathfrak{Br}\,(X/PGL_n)$.

It is important to note that $\mathfrak{Br}\,(X/PGL_n)$ depends on the action of PGL_n on X and not just on the quotient space $Y = X/PGL_n$. Note also that the bundle of central simple algebras $B = M_n(k) \times^{PGL_n} X$ over Y is essentially equivalent to the PGL_n action on X, because X can be recovered as the Y-scheme that represents the functor of isomorphisms between B and the trivial bundle of central simple algebras $M_n(k) \times Y$ over Y. The PGL_n action is recovered via its action on $M_n(k)$. Moreover, we have the following.

Lemma 3.2. *Let PGL_n act freely on affine algebraic varieties X_1 and X_2. Let*

$$\phi\colon X_1/PGL_n \to X_2/PGL_n$$

be a dominant rational map. Then there is a PGL_n-equivariant dominant rational map $\Phi\colon X_1 \to X_2$ making the following diagram commute

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi} & X_2 \\ \downarrow & & \downarrow \\ X_1/PGL_n & \xrightarrow{\phi} & X_2/PGL_n \end{array}$$

if and only if $\mathfrak{Br}\,(X_1/PGL_n) = \phi^{-1}\mathfrak{Br}\,(X_2/PGL_n)$.

Proof. After restricting to suitable open subvarieties and taking the pullback along ϕ we may assume that ϕ is the identity map. We have two distinct PGL_n bundles. These have the same Brauer class if and only if over a suitable open subvariety of X/PGL_n the associated bundles of central simple algebras are isomorphic or equivalently the two PGL_n bundles are isomorphic over this open subvariety. $\square$

We can now define the Brauer classes on (the function fields of) our moduli spaces that we will use in the rest of the paper. For each type (r, d), fix one vector bundle F, which is general in the sense of Proposition 2.3 and recall that $h = \mathrm{hcf}(r, d)$. Let $X_{r,d}$ be the open subset of $\textsc{Quot}(F^h, r, d)$, which parametrizes (up to scaling) quotients $q\colon F^h \to E$ of type (r, d) which are stable bundles and for which the induced map $k^h \to \mathrm{Hom}(F, E)$ is an isomorphism. The obvious action of $GL_h = \mathrm{Aut}(F^h)$ induces a free action of PGL_h on $X_{r,d}$ and the map $X_{r,d} \to \mathfrak{M}_{r,d}$, which forgets the quotient map q, identifies $X_{r,d}/PGL_h$ with an open dense subset of $\mathfrak{M}_{r,d}$ and, in particular, identifies their function fields. Since $\mathfrak{M}_{r,d}$ is a projective variety we may replace $X_{r,d}$ by an open dense affine PGL_h-equivariant subset of itself by

taking the inverse image of some open dense affine subset of $\mathfrak{M}_{r,d}$ contained in the image of $X_{r,d}$.

Definition 3.3. For every type (r, d), the Brauer class $\psi_{r,d}$ on $\mathfrak{M}_{r,d}$ is the class corresponding to $\mathfrak{Br}\,(X_{r,d}/PGL_h)$ after we identify $k(X_{r,d}/PGL_h)$ with $k(\mathfrak{M}_{r,d})$ as described above.

There are more general Brauer classes that arise naturally on X/PGL_n, which we now describe and relate to $\mathfrak{Br}\,(X/PGL_n)$. Let P be a vector bundle over the algebraic variety X on which GL_n acts lifting the action of PGL_n on X such that k^* acts with weight w on the fibres of P. We will call such a bundle P a vector bundle of weight w on X; the GL_n action on P lifting the PGL_n action on X will be implicit. If P is a vector bundle of weight 0, then PGL_n acts on P and P/PGL_n is a vector bundle over X/PGL_n. If P is a vector bundle of weight w then $P^\vee \otimes P$ is a vector bundle of weight 0 and $P^\vee \otimes P/PGL_n$ is a bundle of central simple algebras over X/PGL_n. The bundle of central simple algebras associated to the PGL_n action of X is the special case where P is taken to be the bundle of weight 1 over X given by $k^n \times X$, with GL_n acting diagonally, since we may identify $M_n(k)$ with $(k^n)^\vee \otimes k^n$. We define the Brauer class defined by P to be the Brauer class of the central simple algebra over $k(X/PGL_n)$ defined by the generic fibre of the bundle of central simple algebras $P^\vee \otimes P/PGL_n$.

Lemma 3.4. *Let P be a vector bundle of weight w over an algebraic variety X on which PGL_n acts freely. Then the Brauer class defined by P is $w\mathfrak{Br}\,(X/PGL_n)$.*

Proof. Let P and Q be vector bundles of weight w. Then $P^\vee \otimes Q$ is a vector bundle of weight 0 and $P^\vee \otimes Q/PGL_n$ is a vector bundle over X/PGL_n. It has a structure of a bimodule with $P^\vee \otimes P/PGL_n$ acting on the left and $Q^\vee \otimes Q/PGL_n$ acting on the right. Over the generic point of X/PGL_n it defines a Morita equivalence between (the generic fibres of) $P^\vee \otimes P/PGL_n$ and $Q^\vee \otimes Q/PGL_n$. Hence the Brauer classes defined by P and Q are equal; in other words the Brauer class depends only on the weight.

Now, if $w > 0$, then $Q_w = (k^n)^{\otimes w} \times X$ with the diagonal action of GL_n is a vector bundle of weight w and $Q_w^\vee \otimes Q_w/PGL_n$ is the wth tensor power of $Q_1^\vee \otimes Q_1/PGL_n$. Since the class defined by Q_1 is $\mathfrak{Br}\,(X/PGL_n)$, the class defined by Q_w is $w\mathfrak{Br}\,(X/PGL_n)$. On the other hand, $Q_{-w} = ((k^n)^\vee)^{\otimes w} \times X$ with the diagonal action of GL_n is a vector bundle of weight

$-w$. In particular, $Q_{-1}^{\vee} \otimes Q_{-1}/PGL_n$ is the sheaf of algebras opposite to $Q_1^{\vee} \otimes Q_1/PGL_n$ and therefore the class defined by Q_{-1} is $-\mathfrak{Br}\,(X/PGL_n)$ and, as above, the class defined by Q_{-w} is $-w\mathfrak{Br}\,(X/PGL_n)$. Finally, $\mathcal{O}_X$ is a vector bundle of weight 0 and defines the class 0. $\square$

Thus, if P is a vector bundle of weight 1 and rank r, then the Brauer class $\mathfrak{Br}\,(X/PGL_n)$ is represented by a central simple algebra of dimension r^2, namely $P^{\vee} \otimes P/PGL_n$. It will be important to observe that, birationally, the converse is true. More precisely, we have the following.

Lemma 3.5. *Let PGL_n act freely on an algebraic variety X and suppose that the Brauer class $w\mathfrak{Br}\,(X/PGL_n)$ is represented by a central simple algebra S of dimension s^2 over $k(X/PGL_n)$. Then there exists a PGL_n-equivariant open subset Y of X and a vector bundle Q of weight w over Y whose rank is s.*

Proof. Let P be a vector bundle of weight w and rank p. It is enough to deal with the case where S is a division algebra since the remaining cases are all matrices over this and hence the values for s that arise are all multiples of this. In particular, therefore, we may assume that s divides p. If $s = p$, there is nothing to prove so we may assume that $s < p$. Thus at the generic point of $P^{\vee} \otimes P/PGL_n$, there is an idempotent of rank s. This idempotent is defined over some open subset of X which is PGL_n-equivariant since the idempotent is PGL_n-invariant and gives a decomposition $P \cong P_1 \oplus P_2$ as a direct sum of vector bundles which are GL_n-equivariant one of which has rank s. These bundles have weight w since they are subbundles of P which has weight w. $\square$

We now come to the main object of this section, to describe the relationship between the Brauer classes considered above and twisted Grassmannian bundles such as $\lambda_F \colon \mathfrak{M}_{r,d} \to \mathfrak{M}_{r_1,d_1}$. We start in the general context of Grassmannian bundles associated to a vector bundle P of weight w, although in the end we will only need to consider weights ± 1. Let $j < \mathrm{rk}(P)$ be a positive integer. Then PGL_n acts freely on the bundle of Grassmannians $\mathrm{Gr}(j, P)$ over X and $\phi\colon \mathrm{Gr}(j,P)/PGL_n \to X/PGL_n$ is a Grassmannian bundle over X/PGL_n that is usually not trivial in the Zariski topology. Since the map from $\mathrm{Gr}(j,P)$ to X is PGL_n-equivariant the Brauer class $\mathfrak{Br}\,(\mathrm{Gr}(j,P)/PGL_n)$ is just the pullback of the Brauer class $\mathfrak{Br}\,(X/PGL_n)$. We can also realise the algebraic variety $\mathrm{Gr}(j,P)/PGL_n$ as a quotient variety for a free action of the algebraic group PGL_j on the partial frame bundle

of j linearly independent sections of the vector bundle P and we will need to know how to relate the two Brauer classes we obtain in this way.

We must take care to differentiate two distinct ways of constructing the partial frame bundle. Let S be the universal sub-bundle on $\mathrm{Gr}(j, P)$. Let $\mathrm{Fr}(j, P)$ be the 'covariant' partial frame bundle, whose fibre at x consists of isomorphisms $k^j \to S_x$ and let $\mathrm{Fr}^\vee(j, P)$ be the 'contravariant' partial frame bundle, whose fibre at x consists of isomorphisms $(k^j)^\vee \to S_x$. Then GL_j acts freely on both $\mathrm{Fr}(j, P)$ and $\mathrm{Fr}^\vee(j, P)$ and the quotient variety is $\mathrm{Gr}(j, P)$ in both cases. The difference is that the pullback of S to $\mathrm{Fr}(j, P)$ is the trivial bundle with fibre k^j on which GL_j acts with weight 1, while the pullback of S to $\mathrm{Fr}^\vee(j, P)$ is the trivial bundle with fibre $(k^j)^\vee$ on which GL_j acts with weight -1. The obvious isomorphism between the two frame bundles is compatible with the transpose inverse automorphism of GL_j, but not with the identity automorphism.

The action of GL_n lifts from $\mathrm{Gr}(j, P)$ to $\mathrm{Fr}(j, P)$ and $\mathrm{Fr}^\vee(j, P)$, so both carry an action of $GL_j \times GL_n$. The kernel of each action is isomorphic to k^*, but in the covariant case it is $\{(t^w I, tI) : t \in k^*\}$, while in the contravariant case it is $\{(t^w I, t^{-1} I) : t \in k^*\}$. (Recall that w is the weight of the action of GL_n on P.) Hence, both $\mathrm{Fr}(j, P)/GL_n$ and $\mathrm{Fr}^\vee(j, P)/GL_n$ carry free actions of PGL_j which determine Brauer classes on the quotient, which is equal to $\mathrm{Gr}(j, P)/PGL_n$ in both cases.

We summarise the maps considered above in the following commutative diagram for the case of the covariant partial frame bundle. Note that the groups that appear as labels below the arrows indicate that the maps are quotient maps by a (generically) free action of the group.

$$\begin{tikzcd}
 & & \mathrm{Fr}(j,P) \arrow[dl, "GL_j"] \arrow[dr, "GL_n"] & \\
 & \mathrm{Gr}(j,P) \arrow[dl] \arrow[dr, "PGL_n"'] & & \mathrm{Fr}(j,P)/GL_n \arrow[dl, "PGL_j"] \\
X \arrow[dr, "PGL_n"'] & & \mathrm{Gr}(j,P)/PGLn \arrow[dl, "\phi"'] & \\
 & X/PGL_n & &
\end{tikzcd} \tag{3.1}$$

The diagram for the contravariant frame bundle is of the identical form, but the need to distinguish the two cases is made clear by the following result, which describes the relationship between the Brauer classes determined by all the PGL actions in the diagram.

Lemma 3.6. *Let P be a bundle of weight w on X. Then, in the notation described above,*

$$\begin{aligned}\mathfrak{Br}\left(\mathrm{Gr}(j,P)/PGL_n\right) &= \phi^*\left(\mathfrak{Br}\left(X/PGL_n\right)\right)\\ \mathfrak{Br}\left((\mathrm{Fr}(j,P)/GL_n)/PGL_j\right) &= w\mathfrak{Br}\left(\mathrm{Gr}(j,P)/PGL_n\right)\\ \mathfrak{Br}\left((\mathrm{Fr}^\vee(j,P)/GL_n)/PGL_j\right) &= -w\mathfrak{Br}\left(\mathrm{Gr}(j,P)/PGL_n\right)\end{aligned}$$

Proof. The first equality follows immediately from the fact that the lower diamond in (3.1) is an equivariant pullback. The action of GL_n on P over X lifts naturally to an action of GL_n on the universal subbundle S over $\mathrm{Gr}(j,P)$. Hence S has weight w and so the Brauer class on $\mathrm{Gr}(j,P)/PGL_n$ represented by $S^\vee \otimes S/PGL_n$ is $w\mathfrak{Br}\left(\mathrm{Gr}(j,P)/PGL_n\right)$ by Lemma 3.4. As already observed, the pullback S' of S to $\mathrm{Fr}(j,P)$ is trivial with fibre k^j and so the quotient by GL_n also gives a trivial bundle S'' with fibre k^j on $\mathrm{Fr}(j,P)/GL_n$. But then $\mathfrak{Br}\left((\mathrm{Fr}(j,P)/GL_n)/PGL_j\right)$ is equal to the Brauer class represented by $(S'')^\vee \otimes (S'')/PGL_j$, which is equal to the Brauer class represented by $S^\vee \otimes S/PGL_n$, completing the proof in the covariant case. The proof in the contravariant case is identical except that now S' and S'' are trivial with fibre $(k^j)^\vee$ so that $\mathfrak{Br}\left((\mathrm{Fr}^\vee(j,P)/GL_n)/PGL_j\right)$ is the negative of the class represented by $(S'')^\vee \otimes (S'')/PGL_j$. ☐

We may now describe the rational map $\lambda_F\colon \mathfrak{M}_{r,d} \to \mathfrak{M}_{r_1,d_1}$ as a Grassmannian bundle of the type described above and determine the behaviour of the Brauer classes under pullback. Recall that there exists a vector bundle F_1 and an open subset X_{r_1,d_1} of $\mathrm{Quot}(F_1^{h_1}, r_1, d_1)$ such that the map to $\mathfrak{M}_{r_1,d_1}$ is birational to the PGL_{h_1} quotient map.

Proposition 3.7. *On an open subset of X_{r_1,d_1}, there exists a vector bundle P of weight -1 and of rank lh_1 for some integer l such that the rational map*

$$\lambda_F\colon \mathfrak{M}_{r,d} \to \mathfrak{M}_{r_1,d_1}$$

is birational to the Grassmannian bundle

$$\phi : \mathrm{Gr}(h,P)/PGL_{h_1} \to X_{r_1,d_1}/PGL_{h_1}.$$

Furthermore,

$$\lambda_F^*(\psi_{r_1,d_1}) = \psi_{r,d}.$$

Proof. The idea of the proof is that since F is general, the set of quotients $p : F \otimes V \to E \in X_{r,d}$ such that $E_1 := \ker p \in \mathfrak{M}_{r_1,d_1}$, $Ext(F,E) = 0$ and $Ext(E_1,F) = 0$ is not empty. It is bijective to the set of h-dimensional subspaces $V^\vee \subset Hom(E_1,F)$ such that $p : E_1 \to F \otimes V$ is injective, $E := \operatorname{coker} p \in \mathfrak{M}_{r,d}$, $Ext(E_1,F) = 0$ and $Ext(F,E) = 0$. We fill in the details below.

Consider the open set in X_{r_1,d_1} parametrizing those $q_1 \colon F_1^{h_1} \to E_1$ for which $\mathrm{Ext}(E_1,F) = 0$. Over this open set there is a vector bundle P whose fibre at $[q_1]$ is $\mathrm{Hom}(E_1,F)$. Since PGL_{h_1} acts with weight 1 on E_1, it acts with weight -1 on P. We claim that $\mathrm{Gr}(h,P)/PGL_{h_1}$ is birational to $\mathfrak{M}_{r,d}$. To see this, consider the contravariant partial frame bundle $\mathrm{Fr}^\vee(h,P)$ whose fibre over $[q_1] \in X_{r_1,d_1}$ is naturally identified with the set of maps $p \colon E_1 \to F^h$ for which the induced map $(k^h)^\vee \to \mathrm{Hom}(E_1,F)$ is injective. On an open subset in $\mathrm{Fr}^\vee(h,P)$, the map p is injective as a map of bundles and its cokernel $q : F^h \to E$ gives a point in $X_{r,d}$. Since p, but not q_1, is determined by q we see that $\mathrm{Fr}^\vee(h,P)/GL_{h_1}$ is birational to $X_{r,d}$ and so $(\mathrm{Fr}^\vee(h,P)/GL_{h_1})/PGL_h$ is birational to $\mathfrak{M}_{r,d}$. Since $\mathrm{Gr}(h,P)$ is $\mathrm{Fr}(h,P)/GL_h$, we deduce that $\mathrm{Gr}(h,P)/PGL_{h_1}$ is birational to $\mathfrak{M}_{r,d}$.

We can arrange all the rational maps we have considered above into the following diagram of the form of (3.1).

$$\begin{tikzcd}
 & & \mathrm{Fr}^\vee(h,P) \arrow[dl, "GL_h"'] \arrow[dr, "GL_{h_1}"] & \\
 & \mathrm{Gr}(h,P) \arrow[dl] \arrow[dr, "PGL_{h_1}"'] & & X_{r,d} \arrow[dl, "PGL_h"] \\
X_{r_1,d_1} \arrow[dr, "PGL_{h_1}"'] & & \mathfrak{M}_{r,d} \arrow[dl, "\lambda_F"'] & \\
 & \mathfrak{M}_{r_1,d_1} & &
\end{tikzcd} \tag{3.2}$$

Since P has weight -1, the first and last formulae in Lemma 3.6 give

$$\lambda_F^*(\psi_{r_1,d_1}) = \mathfrak{Br}\left(\mathrm{Gr}(h,P)/PGL_{h_1}\right) = \psi_{r,d}$$

which completes the proof. $\square$

4 The Hecke correspondence

One of the main ideas of the paper is to compare the (birationally) twisted Grassmannian bundle $\lambda_F\colon \mathfrak{M}_{r,d} \to \mathfrak{M}_{r_1,d_1}$ to another Grassmannian bundle which is twisted by the same amount but has smaller fibres. This second bundle is provided by the Hecke correspondence, which we describe in this section. Within this section, we may let h and h_1 be arbitrary integers with $h \le h_1$. Only later, will we need to use the fact that h actually divides h_1.

Let $\mathfrak{P}_{h_1,0,h}$ be the moduli space of parabolic bundles, which parametrizes pairs consisting of a bundle (or locally free sheaf) $\mathcal{E}_1$ of type $(h_1, 0)$ together with a locally free subsheaf $\mathcal{E}_2 \subset \mathcal{E}_1$ such that the quotient $\mathcal{E}_1/\mathcal{E}_2$ is isomorphic to $\big(\mathcal{O}_x\big)^h$ for a fixed point $x \in C$. In order to specify a projective moduli space exactly, we would need to specify parabolic weights to determine notions of stability and semistability. However, we are only interested in this space up to birational equivalence and it is known ([1] Section 4) that the birational type of the moduli space does not depend on the choice of parabolic weights. Indeed, we may choose to let $\mathfrak{P}_{h_1,0,h}$ denote the dense open set of quasi-parabolic bundles $\mathcal{E}_2 \subset \mathcal{E}_1$ that are stable for all choices of parabolic weights.

The type of $\mathcal{E}_2$ must be $(h_1, -h)$ and there are two dominant rational maps

$$\theta_1\colon \mathfrak{P}_{h_1,0,h} \dashrightarrow \mathfrak{M}_{h_1,0}\colon [\mathcal{E}_2 \subset \mathcal{E}_1] \mapsto [\mathcal{E}_1]$$
$$\theta_2\colon \mathfrak{P}_{h_1,0,h} \dashrightarrow \mathfrak{M}_{h_1,-h}\colon [\mathcal{E}_2 \subset \mathcal{E}_1] \mapsto [\mathcal{E}_2]$$

The key point is that, like λ_F, the maps θ_1 and θ_2 are (birational to) twisted Grassmannian bundles whose twisting is measured by the Brauer classes $\psi_{h_1,0}$ and $\psi_{h_1,-h}$ respectively. Furthermore, as we shall show below, these two Brauer classes pull back to the same class on $\mathfrak{P}_{h_1,0,h}$.

To construct $\mathfrak{P}_{h_1,0,h}$ birationally from $\mathfrak{M}_{h_1,0}$, let H_1 be the vector bundle over $X_{h_1,0}$ whose fibre over the point $\big[q_1 : F_1^{h_1} \to E_1\big]$ is $\mathrm{Hom}(E_1, \mathcal{O}_x)$, where $\mathcal{O}_x$ is the structure sheaf of the point $x \in C$. Then H_1 is a vector bundle of weight -1 and $\mathfrak{P}_{h_1,0,h}$ is birational to $\mathrm{G{\scriptstyle R}}(h, H_1)/PGL_{h_1}$. To see this, consider the contravariant frame bundle $\mathrm{F{\scriptstyle R}}^\vee(h, H_1)$. A point in the fibre over $[q_1]$ may be identified with a map

$$p\colon E_1 \to (\mathcal{O}_x)^h \tag{4.1}$$

such that the induced map $(k^h)^\vee \to \mathrm{Hom}(E, \mathcal{O}_x)$ is injective. If we restrict to the open set on which p is also surjective so that it determines a quasi-parabolic structure, then the map to $\mathfrak{P}_{h_1,0,h}$ which forgets p and q_1 is precisely the quotient by GL_h that gives $\mathrm{Gr}(h, H_1)$, followed by the quotient by PGL_{h_1}.

Thus we have another diagram of the form of (3.1).

$$\begin{tikzcd}
 & & \mathrm{Fr}^\vee(h, H_1) \arrow[dl, "GL_h"] \arrow[dr, "GL_{h_1}"] & \\
 & \mathrm{Gr}(h, H_1) \arrow[dl] \arrow[dr, "PGL_{h_1}"'] & & \mathrm{Fr}^\vee(h, H_1)/GL_{h_1} \arrow[dl, "PGL_h"] \\
X_{h_1,0} \arrow[dr, "PGL_{h_1}"'] & & \mathfrak{P}_{h_1,0,h} \arrow[dl, "\theta_1"'] & \\
 & \mathfrak{M}_{h_1,0} & &
\end{tikzcd} \tag{4.2}$$

Hence, by Lemma 3.6, we have

$$\theta_1^*(\psi_{h_1,0}) = \mathfrak{Br}\left((\mathrm{Fr}^\vee(h, H_1)/GL_{h_1})/PGL_h\right) \tag{4.3}$$

Now we construct $\mathfrak{P}_{h_1,0,h}$ birationally from $\mathfrak{M}_{h_1,-h}$. To preserve the generality of this section, let $m = \mathrm{hcf}(h_1, h)$, but note that $m = h$ in the case of real interest. Let H_2 be the vector bundle over $X_{h_1,-h}$ whose fibre above a point $[q_2 : F_2^m \to E_2]$ is $\mathrm{Ext}(\mathcal{O}_x, E_2)$. Then H_2 has weight 1 and $\mathfrak{P}_{h_1,0,h}$ is also birational to $\mathrm{Gr}(h, H_2)/PGL_m$. This follows, as above, by considering the open set in $\mathrm{Fr}(h, H_2)$ parametrizing extensions

$$0 \to E_2 \to E_1 \to (\mathcal{O}_x)^h \to 0. \tag{4.4}$$

such that the induced map $k^h \to \mathrm{Ext}(\mathcal{O}_x, E_2)$ is injective. The moduli space $\mathfrak{P}_{h_1,0,h}$ arises (birationally) by taking the quotient by GL_h and then PGL_m.

Thus, again, we have a diagram of the form of (3.1).

$$\begin{tikzcd}
 & & \mathrm{F{\scriptsize R}}(h, H_2) \arrow[dl, "GL_h"'] \arrow[dr, "GL_m"] & \\
 & \mathrm{G{\scriptsize R}}(h, H_2) \arrow[dl] \arrow[dr, "PGL_m"'] & & \mathrm{F{\scriptsize R}}(h, H_2)/GL_m \arrow[dl, "PGL_h"] \\
X_{h_1,-h} \arrow[dr, "PGL_m"'] & & \mathfrak{P}_{h_1,0,h} \arrow[dl, "\theta_2"'] & \\
 & \mathfrak{M}_{h_1,-h} & &
\end{tikzcd} \tag{4.5}$$

and, by Lemma 3.6, we have

$$\theta_2^*(\psi_{h_1,-h}) = \mathfrak{Br}\left((\mathrm{F{\scriptsize R}}(h, H_2)/GL_m)/PGL_h\right) \tag{4.6}$$

But now we simply need to observe that the data in (4.1) and in (4.4) have the same form and differ only in the imposition of different open conditions. Thus we may identify open subsets of $\mathrm{F{\scriptsize R}}(h, H_2)/GL_m$ and $\mathrm{F{\scriptsize R}}(h, H_1)/GL_{h_1}$. One could in principle identify both of these as open subsets of an appropriate fine moduli space for such data. Furthermore, this identification is compatible with the PGL_h actions and so we may combine (4.3) and (4.6) to obtain

$$\theta_2^*(\psi_{h_1,-h}) = \theta_1^*(\psi_{h_1,0}). \tag{4.7}$$

5 Construction of birationally linear maps

We may now proceed with the proof of the main theorem of the paper on the existence of a birationally linear map from the moduli space $\mathfrak{M}_{r,d}$ of vector bundles of rank r and degree d to the moduli space $\mathfrak{M}_{h,0}$. The proof goes by induction on the stronger statement that there is such a birationally linear rational map that preserves the Brauer classes defined in the Section 3. More precisely, we prove the following theorem.

Theorem 5.1. *Let $\psi_{r,d}$ be the Brauer class on $\mathfrak{M}_{r,d}$ defined for every type (r, d) in Definition 3.3 and let $h = \mathrm{hcf}(r, d)$. Then there exists a birationally linear map $\mu\colon \mathfrak{M}_{r,d} \dashrightarrow \mathfrak{M}_{h,0}$ such that $\mu^*(\psi_{h,0}) = \psi_{r,d}$.*

Proof. If r divides d, then $h = r$ and $\mathfrak{M}_{r,d}$ is isomorphic to $\mathfrak{M}_{h,0}$ by tensoring with a line bundle of degree d/r. This isomorphism may be taken to be μ and it preserves the Brauer class. Otherwise, we saw in Section 2 how to construct a map $\lambda_F \colon \mathfrak{M}_{r,d} \dashrightarrow \mathfrak{M}_{r_1,d_1}$ with $r_1 < r$ and we proved in Proposition 3.7 that

$$\lambda_F^*(\psi_{r_1,d_1}) = \psi_{r,d}. \tag{5.1}$$

We construct the map μ, by induction on the rank r, as the composite of the top row of the following commutative diagram of dominant rational maps which combines λ_F and the Hecke correspondence described in Section 4. The other elements of the diagram we will explain next.

$$\begin{tikzcd}
\mathfrak{M}_{r,d} \arrow[r, "\rho"] \arrow[dr, "\lambda_F"'] & \widehat{\mathfrak{P}} \arrow[r, "\widehat{\mu}_1"] \arrow[d, "\widehat{\theta}_1"] & \mathfrak{P}_{h_1,0,h} \arrow[r, "\theta_2"] \arrow[d, "\theta_1"] & \mathfrak{M}_{h_1,-h} \arrow[r, "\mu_2"] & \mathfrak{M}_{h,0} \\
 & \mathfrak{M}_{r_1,d_1} \arrow[r, "\mu_1"] & \mathfrak{M}_{h_1,0} & &
\end{tikzcd} \tag{5.2}$$

The maps μ_1 and μ_2 are of the same sort as μ and may be assumed to exist by induction, since both r_1 and $h_1 = \mathrm{hcf}(r_1, d_1)$ are less than r. Thus they are birationally linear and satisfy

$$\mu_1^*(\psi_{h_1,0}) = \psi_{r_1,d_1} \tag{5.3}$$
$$\mu_2^*(\psi_{h,0}) = \psi_{h_1,-h}. \tag{5.4}$$

because $h = \mathrm{hcf}(h_1, -h)$ by Lemma 2.4.

The central square in the diagram is a pull back. In particular, $\widehat{\theta}_1 \colon \widehat{\mathfrak{P}} \dashrightarrow \mathfrak{M}_{r_1,d_1}$ is the pull back of θ_1 along μ_1 and hence, by (5.3), it is a Grassmannian bundle over $\mathfrak{M}_{r_1,d_1}$ whose twisting is measured by ψ_{r_1,d_1}. Thus $\widehat{\theta}_1$ and λ_F are twisted Grassmannian bundles associated to vector bundles of weight -1 over X_{r_1,d_1} and of ranks h_1 and lh_1 respectively (see Proposition 3.7). We will prove in Lemma 5.3 below that this implies that there is a birationally linear map $\rho \colon \mathfrak{M}_{r,d} \dashrightarrow \widehat{\mathfrak{P}}$ such that $\lambda_F = \widehat{\theta}_1 \rho$ and hence

$$\rho^* \left(\widehat{\theta}_1^* \left(\psi_{r_1,d_1} \right) \right) = \psi_{r,d}. \tag{5.5}$$

The pullback $\widehat{\mu}_1$ of μ_1 along θ_1 is birationally linear and satisfies

$$\widehat{\mu}_1^* \left(\theta_1^*(\psi_{h_1,0})\right) = \widehat{\theta}_1^* \left(\mu_1^*(\psi_{h_1,0}\right) = \widehat{\theta}_1^* \left(\psi_{r_1,d_1}\right). \tag{5.6}$$

Thus $\widehat{\mu}_1\rho\colon \mathfrak{M}_{r,d} \dashrightarrow \mathfrak{P}_{h_1,0,h}$ is birationally linear and pulls back $\theta_1^*(\psi_{h_1,0})$ to $\psi_{r,d}$. But by (4.7) and (5.4), this means that $\mu = \mu_2\theta_2\widehat{\mu}_1\rho$ pulls back $\psi_{h,0}$ to $\psi_{r,d}$ as required and to complete the proof we need to show that θ_2 is birationally linear. This follows from Lemma 5.2 below, because, as we saw in Section 4, θ_2 is a twisted Grassmannian bundle of h-dimensional subspaces of a vector bundle H_2 of weight 1 over $X_{h_1,-h}$ and the Brauer class $\psi_{h_1,-h}$ is represented by a central simple algebra of dimension h^2. Thus although θ_2 is not locally trivial in the Zariski topology when $h \neq 1$, we can show that it is birationally linear since its generic fibre is birational to a Grassmannian over a division algebra; this is not the way it is expressed in Lemma 5.2 though the translation to this is fairly simple. $\square$

Thus (modulo two lemmas) we have proved Theorem 1.1 as we set out to do. To deduce Theorem 1.2, it is sufficient to observe that, by Lemma 2.4, the map λ_F restricts to a map between moduli spaces of fixed determinant and that the Hecke correspondence restricts to a correspondence between moduli spaces of fixed determinant. Therefore the map $\mu\colon \mathfrak{M}_{r,d} \to \mathfrak{M}_{h,0}$ restricts to a map between fixed determinant moduli spaces, although precisely how the determinants are related will depend on various choices made in the construction. In the case $h = 1$, the fixed determinant moduli space is a point and we obtain Theorem 1.2.

We finish with the proofs of the two lemmas about birationally linear maps that we used in the proof of Theorem 5.1. The first thing we need to understand is when a twisted Grassmannian bundle is birationally linear over its base. We shall provide a sufficient condition which is in fact necessary though we shall not prove that here since we do not need it.

Start by observing that, if P and Q are vector bundles of weight w over X, then $M = P^\vee \otimes Q/PGL_n$ is a bundle of left modules for $P^\vee \otimes P/PGL_n$ and that this correspondence is invertible because $Q = P \otimes_{P^\vee \otimes P} \gamma^* M$, where $\gamma\colon X \to X/PGL_n$ is the quotient map.

Lemma 5.2. *Let P be a vector bundle of weight w over X. Assume that the Brauer class associated to P is represented by a central simple algebra of dimension j^2. Then $\pi\colon \mathrm{G{\scriptsize R}}(j,P)/PGL_n \to X/PGL_n$ is a birationally linear map.*

Proof. Let A be the central simple algebra given by the bundle of central simple algebras $P^\vee \otimes P/PGL_n$ over the field $k(X/PGL_n)$. Then by assumption A has a left ideal of dimension $j\,\mathrm{rk}(P)$ which is of necessity a direct summand of A. Therefore, over some dense open subvariety of X/PGL_n, $P^\vee \otimes P/PGL_n \cong L_1 \oplus L_2$ where L_1 and L_2 are bundles of left ideals for $P^\vee \otimes P/PGL_n$ and $\mathrm{rk}(L_1) = j\,\mathrm{rk}(P)$. We may as well assume that this happens over X/PGL_n. We obtain a corresponding direct sum decomposition of P, $P \cong P_1 \oplus P_2$ where P_1 and P_2 are GL_n stable subbundles of P and hence both of weight w. Also $\mathrm{rk}(P_1) = j$. Now consider the vector bundle $P_1^\vee \otimes P_2$. Let $\lambda\colon P_1 \to P_2$ be the universal homomorphism of vector bundles defined on $P_1^\vee \otimes P_2$ and consider the map of vector bundles over $P_1^\vee \otimes P_2$, $(Id, \lambda)\colon P_1 \to P_1 \oplus P_2 \cong P$. This representation of P_1 as a subbundle of P defines a map from $P_1^\vee \otimes P_2$ to $\mathrm{G{\scriptstyle R}}(j, P)$ which is PGL_n-equivariant, injective and onto an open subvariety of $\mathrm{G{\scriptstyle R}}(j, P)$. Hence $P_1^\vee \otimes P_2/PGL_n$ which is a vector bundle over X/PGL_n is an open subvariety of $\mathrm{G{\scriptstyle R}}(j, P)$. $\square$

It remains to show that two twisted Grassmannian bundles of equal dimensional subspaces arising from vector bundles of the same weight have a birationally linear map between them.

Lemma 5.3. *Let P and Q be vector bundles of weight w over X and suppose that $j < \mathrm{rk}(Q) < \mathrm{rk}(P)$. Then there is a birationally linear rational map*

$$\rho\colon \mathrm{G{\scriptstyle R}}(j, P)/PGL_n \to \mathrm{G{\scriptstyle R}}(j, Q)/PGL_n.$$

compatible with the bundle maps to X/PGL_n.

Proof. $P^\vee \otimes Q/PGL_n$ is a bundle of left modules for $P^\vee \otimes P/PGL_n$ of rank equal to $\mathrm{rk}(P)\,\mathrm{rk}(Q)$ and since $\mathrm{rk}(Q) < \mathrm{rk}(P)$ there is an open subvariety of X/PGL_n on which

$$P^\vee \otimes P/PGL_n \cong P^\vee \otimes Q/PGL_n \oplus L$$

for some vector bundle of left ideals L since this is true at the generic point of X/PGL_n. Hence we may assume that on X/PGL_n, $P \cong Q \oplus Q'$ for GL_n stable subbundles Q and Q'. Let S be the universal subbundle on $\mathrm{G{\scriptstyle R}}(j, Q)$. Then S and Q' are both vector bundles of weight w on $\mathrm{G{\scriptstyle R}}(j, Q)$. We consider the vector bundle $S^\vee \otimes Q'$ over $\mathrm{G{\scriptstyle R}}(j, Q)$. Let $\lambda\colon S \to Q'$ be the universal homomorphism of vector bundles defined on $S^\vee \otimes Q'$ and let $\iota\colon S \to Q$ be

the universal inclusion of S in Q pulled back to $S^\vee \otimes Q'$; now consider the map of vector bundles

$$(\iota, \lambda)\colon S \to Q \oplus Q' \cong P$$

defined on $S^\vee \otimes Q'$. This gives a subbundle of P of rank j and hence defines a map from $S^\vee \otimes Q'$ to $\mathrm{G{\scriptstyle R}}(j, P)$. This map is injective and onto an open subvariety of $\mathrm{G{\scriptstyle R}}(j, P)$ and it is also PGL_n-equivariant. Hence $S^\vee \otimes Q'/PGL_n$ is an open subvariety of $\mathrm{G{\scriptstyle R}}(j, P)/PGL_n$. However, $S^\vee \otimes Q'/PGL_n$ is a vector bundle over $\mathrm{G{\scriptstyle R}}(j, Q)/PGL_n$ which gives our lemma. $\square$

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